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[This question paper contains 4 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 4350

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Unique Paper Code : 32351301

Name of the Paper : Theory of Real Functions

Name of the Course : B.Sc. (H) Mathematics
(LOCF)

Semester : III

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any two parts from each question.
3. All questions are compulsory.

1. (a) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A and $f: A \rightarrow \mathbb{R}$, then define limit of function f at c . Use

$\epsilon - \delta$ definition to show that $\lim_{x \rightarrow 2} \frac{1}{1-x} = -1$. (6)

(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

P.T.O.

Show that f has a limit at $x = 0$. Use sequential criterion to show that f does not have a limit at c if $c \neq 0$. (6)

(c) Show that $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x^2}\right)$ does not exist in \mathbb{R} but

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0. \quad (6)$$

2. (a) Let $A \subseteq \mathbb{R}$, let $f: A \rightarrow \mathbb{R}$ and $g: A \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A . Show that if

$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M \text{ then } \lim_{x \rightarrow c} (fg)(x) = LM. \quad (6)$$

(b) Evaluate the limit $\lim_{x \rightarrow 1^+} \frac{x}{x-1}$. (6)

(c) Let $A \subseteq \mathbb{R}$, $f: A \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$. Show that the following conditions are equivalent-

(i) f is continuous at c .

(ii) For every sequence $\langle x_n \rangle$ in A that converges to c , the sequence $\langle f(x_n) \rangle$ converges to $f(c)$. (6)

3. (a) Let $A, B \subseteq \mathbb{R}$ and let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b \in B$,

then show that the composition function $g \circ f: A \rightarrow \mathbb{R}$ is continuous at c . Also, show that the function $f(x) = \cos(1 + x^2)$ is continuous on \mathbb{R} .

(7½)

(b) State and prove Maximum-Minimum Theorem for continuous functions on a closed and bounded interval. (7½)

(c) State Bolzano's Intermediate value theorem. Show that every polynomial of odd degree with real coefficients has at least one real root. (7½)

4. (a) Let $A \subseteq \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$. Show that if f is continuous at $c \in A$ then $|f|$ is continuous at c . Is the converse true? Justify your answer. (6)

(b) Let $I = [a, b]$ and $f: I \rightarrow \mathbb{R}$. Show that if f is continuous on I then it is uniformly continuous on I . (6)

(c) Show that $f(x) = \sin x$ is uniformly continuous on \mathbb{R} and the function $g(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ is not uniformly continuous on $(0, \infty)$. (6)

5. (a) Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions that are differentiable at c . Prove that the function fg is differentiable at c , and $(fg)' = f'(c)g(c) + f(c)g'(c)$. (6)

P.T.O.

(b) Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that g is differentiable for all $x \in \mathbb{R}$.

Also, show that the derivative g' is not continuous at $x = 0$. (6)

(c) Suppose that f is continuous on a closed interval $I = [a, b]$, and that f has a derivative in the open interval (a, b) . Prove that there exists at least one point c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

Suppose that $f: [0, 2] \rightarrow \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ and that $f(0) = 0$, $f(2) = 1$. Show that there exists $c_1 \in (0, 2)$ such that $f'(c_1) = \frac{1}{2}$. (6)

6. (a) Find the points of relative extrema of the function $f(x) = 1 - (x - 1)^{2/3}$, for $0 \leq x \leq 2$. (6)

(b) Let I be an open interval and let $f: I \rightarrow \mathbb{R}$ has a second derivative on I . Then show that f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$. (6)

(c) Obtain Taylor's series expansion for the function $f(x) = \sin x$, $\forall x \in \mathbb{R}$. (6)

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